Quantum Qualifying Exam

January 20, 2021

Short Problem 1

Considering the angular momentum raising and lowering operators $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$, show that:

a) $[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm$

b) $[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z$

c) $[\hat{J}_+, J^2] = 0$

Short Problem 2

A particle is constrained to move in an infinitely deep square potential well, spanning from $0 < x <$ a. Suppose we add a delta function bump in the center of the well to produce the perturbation:

$$H' = \alpha\delta(x - a/2) \tag{1}$$

Where α is a constant. Find the first order correction to the nth allowed value of the energy. Explain why there is no correction for even n.

1 Long Problem 1

The 1D Harmonic oscillator has Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{2}$$

a) By directly substition into the Schrodinger equation, show that one of its stationary states is a Gaussian function. Find its width σ in terms of $m, \omega, \hbar$. Show that the energy of this state is $E_0 = \frac{1}{2}\hbar\omega$.

b) Show using a ladder operator that the state you found from part (a) is the lowest energy state.

c) Find the wave function of the first excited state.

d) Show that the Hamiltonian can be written as:

$$H = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega \tag{3}$$

And thus that the energies of the states of this system are given by:

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \tag{4}$$

You may find it helpful to recall that the ladder operators for the Harmonic oscillator are given by:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \tag{5}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \tag{6}$$

Long Problem 2:

Consider a spin $\frac{1}{2}$ particle in a magnetic field. The Hamiltonian is given in terms of a constant α and the gyromagnetic ratio g by:

$$H_0 = \alpha\vec{B}.(\vec{L} + g\vec{S}) \tag{7}$$

a) The system is handed to us in a state of well defined total angular momentum in the direction of the B field, $(j, m_j) = (\frac{1}{2}, \frac{1}{2})$. There are three possible assignments of spin and orbital angular momentum consistent with this total angular momentum state. Give their wavefunctions in terms of spin states $|\uparrow\rangle, |\downarrow\rangle$ and angular momentum states $|l, m_l\rangle$.

b) Calculate the energy perturbations of the above states caused by the magnetic field.

c) Now we consider the effects of including a spin orbit coupling. Assume β is a constant and $\beta \ll \alpha$:

$$H = H_0 + \beta\vec{L}.\vec{S} \tag{8}$$

calculate shift in energy, to first order in β, for each of the states you found for section (a) due to this perturbation.