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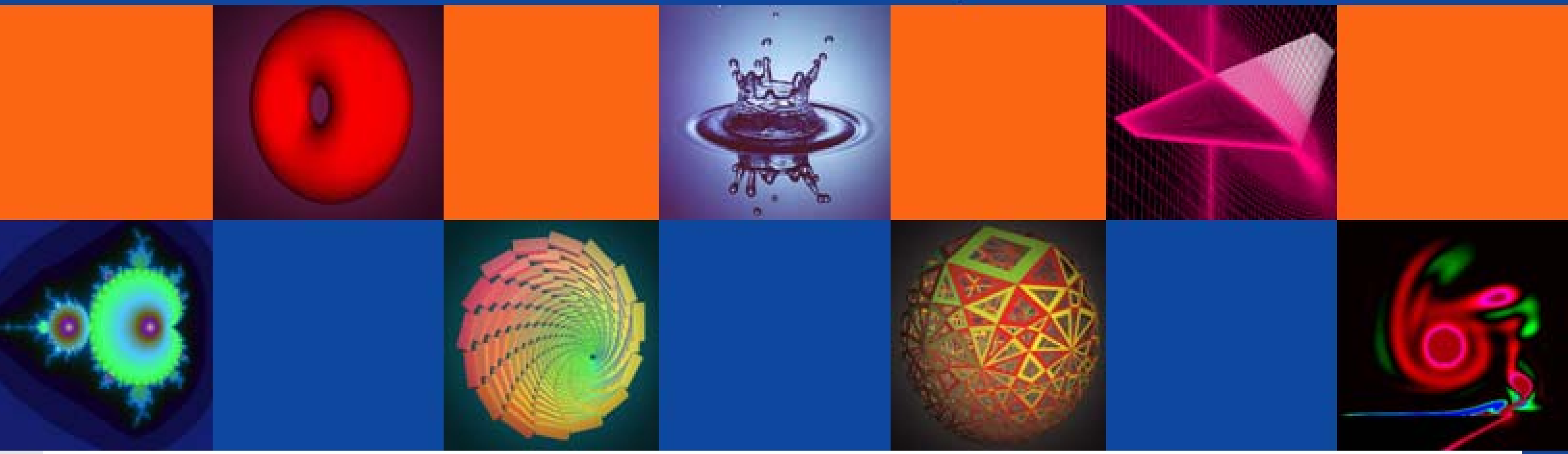

On Acyclic Complexes of Free Modules

Meri T. Hughes,
David A. Jorgensen, and
Liana M. Sega

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ON ACYCLIC COMPLEXES OF FREE MODULES

MERI T. HUGHES, DAVID A. JORGENSEN, AND LIANA M. ŞEGA

Abstract. We consider the question of how minimal acyclic complexes of free modules arise over a commutative local ring. A standard construction gives that every totally reflexive module yields such a complex. We show that for certain rings this construction is essentially the only method of obtaining such complexes. We also give examples of rings which admit acyclic complexes of free modules which cannot be obtained by means of this construction.

INTRODUCTION

Let R be a commutative ring. An *acyclic complex* of projective R-modules is a complex

$$\mathbf{A}: \qquad \cdots \to A_2 \xrightarrow{d_2^{\mathbf{A}}} A_1 \xrightarrow{d_1^{\mathbf{A}}} A_0 \xrightarrow{d_0^{\mathbf{A}}} A_{-1} \xrightarrow{d_{-1}^{\mathbf{A}}} A_{-2} \to \cdots$$

with A_i projective for each i and $\mathrm{H}(\mathbf{A}) = 0$. An acyclic complex of projectives $\mathbf{A}$ satisfying $\mathrm{H}(\mathbf{A}^*) = 0$, where $\mathbf{A}^* = \mathrm{Hom}_R(\mathbf{A}, R)$, is said to be *totally acyclic*, or a *complete resolution*. Such complexes are foundational in the theory of complete (or Tate) (co)homology, and determine important characteristics of the ring R and the category of R-modules, see [4] and [1] for example. Properties and uses of totally acyclic complexes have been studied extensively. More recently, the failure of acyclic complexes to be totally acyclic is studied in [8] and [7]. In this paper we are concerned with the more fundamental question of how acyclic complexes of projectives arise. We show that when certain classical ring invariants are not too small, such complexes can occur apart from a canonical construction involving dualization.

We assume throughout that R is a local ring, with maximal ideal $\mathfrak{m}$, and we assume that our complexes are locally finitely generated. In this case acyclic complexes of finitely generated projective modules are acyclic complexes of finitely generated free modules, and one has a notion of minimality of such complexes. This allows us to ignore trivial constructions, such as split-exact complexes of free modules.

Every R-module M has a free resolution, but it is not always possible to extend this resolution (to the right) to an acyclic complex of free modules $\mathbf{A}$ with $M = \mathrm{Coker}\, d_1^{\mathbf{A}}$. A standard process by which one can extend a resolution goes as follows: if there exists an R-module N such that $M \cong N^*$, where $N^* = \mathrm{Hom}_R(N, R)$, and $\mathrm{Ext}^i_R(N, R) = 0$ for all $i > 0$, then one can splice together a free resolution of M with the dual of a free resolution of N in order to obtain an acyclic complex

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of free R-modules. This construction is described in more detail in Construction 1.1 of Section 1. We shall refer to complexes obtained from this construction as *semi-dualized complexes.*

Every totally acyclic complex of free modules is semi-dualized. Moreover, the known examples of minimal acyclic complexes of finitely generated free modules which are not totally acyclic, constructed by Jorgensen and Şega in [8], are also semi-dualized. Christensen and Veliche [6] initiated a systematic study of acyclic complexes of finitely generated free modules over rings R with $\mathfrak{m}^3 = 0$, and raised the question:

Question. ([6, 3.4]) If $\mathbf{A}$ is a minimal acyclic complex of finitely generated free R-modules, then is $\mathbf{A}$ necessarily a semi-dualized complex?

In this paper we give answers to the question in terms of the following classical invariants of the ring R. The *Loewy length* of R is the integer

$$\ell\ell(R) = \min\{n \geq 0 \,|\, \mathfrak{m}^n \subseteq (\boldsymbol{x}) \text{ for some system of parameters } \boldsymbol{x} \text{ of } R\},$$

and the *codimension* of R, $\operatorname{codim} R$, is the number $\operatorname{edim} R - \dim R$, where $\operatorname{edim} R$ denotes the minimal number of generators of $\mathfrak{m}$.

In Section 1 we show that the question has a positive answer provided $\operatorname{codim} R \leq 2$, or R is Cohen-Macaulay with either $\ell\ell(R) \leq 2$, or $\operatorname{codim} R = \ell\ell(R) = 3$.

In Section 2 we give examples of minimal acyclic complexes of free modules over Cohen-Macaulay rings R with $\operatorname{codim} R \geq 5$ and $\ell\ell(R) \geq 3$ over which the question has a negative answer.

The question remains open for local rings with $\operatorname{codim} R = 4$ and $\ell\ell(R) \geq 3$.

When $\mathfrak{m}^3 = 0$ and R is not Gorenstein, it is shown in [6] that a minimal acyclic complex of finitely generated free R-modules $\mathbf{A}$ can only have one of the following types of behavior:

(i) The sequence $\{\operatorname{rank}_R A_i\}$ is constant; this happens precisely when the residue field of R is not a direct summand of $\operatorname{Coker} d_i^{\mathbf{A}}$ for any i.

(ii) There exists an integer χ such that the sequence $\{\operatorname{rank}_R A_i\}_{i\leq\chi}$ is constant and the sequence $\{\operatorname{rank}_R A_i\}_{i\geq\chi}$ is strictly increasing.

Our basic examples in Section 2 satisfy $\mathfrak{m}^3 = 0$ and exhibit behavior of type (i). Question [6, 3.5] asks if all complexes of type (i) are necessarily totally acyclic. Thus we also answer negatively this question.

1. Semi-dualized Complexes of Free Modules

In this section $(R, \mathfrak{m})$ denotes a commutative local ring, with maximal ideal $\mathfrak{m}$. We consider complexes of finitely generated free R-modules

$$\mathbf{A}: \qquad \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}^{\mathbf{A}}} A_n \xrightarrow{d_n^{\mathbf{A}}} A_{n-1} \longrightarrow \cdots$$

For each integer n, the nth *syzygy module* of $\mathbf{A}$ is $\Omega^n\mathbf{A} = \operatorname{Coker} d_{n+1}^{\mathbf{A}}$. The nth *shift* of $\mathbf{A}$ is the complex $\mathbf{\Sigma}^n\mathbf{A}$ with $(\mathbf{\Sigma}^n\mathbf{A})_i = A_{i-n}$ and $d_i^{\mathbf{\Sigma}^n\mathbf{A}} = (-1)^n d_{i-n}^{\mathbf{A}}$. We write $\mathbf{A}_{\geqslant n}$ for the complex with ith component (respectively, differential) equal to A_i (respectively, $d_i^{\mathbf{A}}$) if $i \geq n$ and 0 if $i < n$. The *trivial complex* is the complex with $A_i = 0$ for all i.

We let $(\ )^*$ denote the dualization functor $\operatorname{Hom}_R(\ , R)$. The *dual complex* of $\mathbf{A}$ is the complex $\mathbf{A}^*$, which has component $(A_{-n})^*$ in degree n, and differentials

$d_n^{\mathbf{A}^*} = (d_{-n+1}^{\mathbf{A}})^*$:

$$\mathbf{A}^* : \quad \cdots \longrightarrow (A_{-n-1})^* \xrightarrow{(d_{-n}^{\mathbf{A}})^*} (A_{-n})^* \xrightarrow{(d_{-n+1}^{\mathbf{A}})^*} (A_{-n+1})^* \longrightarrow \cdots$$

To avoid trivial constructions of acyclic complexes of free modules, such as those built from direct sums of complexes of the form $0 \to R \xrightarrow{1} R \to 0$, we use the following notion of minimality: the complex $\mathbf{A}$ is *minimal* if $d_i^{\mathbf{A}}(A_i) \subseteq \mathfrak{m}A_{i-1}$ for all $i \in \mathbb{Z}$.

In general, non-trivial minimal acyclic complexes of free modules may not exist. A standard situation in which they do is described as follows.

1.1. **Construction.** Suppose that M is an R-module satisfying

$$\operatorname{Ext}_R^i(M, R) = 0 \text{ for all } i > 0. \tag{1.1.1}$$

Let $\mathbf{P} \xrightarrow{\pi} M^*$ be a free resolution of M^*, with $\mathbf{P} : \cdots \longrightarrow P_2 \xrightarrow{d_2^{\mathbf{P}}} P_1 \xrightarrow{d_1^{\mathbf{P}}} P_0 \to 0$, and $\mathbf{Q} \xrightarrow{\eta} M$ be a free resolution of M, with $\mathbf{Q} : \cdots \longrightarrow Q_2 \xrightarrow{d_2^{\mathbf{Q}}} Q_1 \xrightarrow{d_1^{\mathbf{Q}}} Q_0 \to 0$. By condition (1.1.1), the complex

$$0 \longrightarrow M^* \xrightarrow{\eta^*} Q_0^* \xrightarrow{(d_1^{\mathbf{Q}})^*} Q_1^* \xrightarrow{(d_2^{\mathbf{Q}})^*} Q_2^* \longrightarrow \cdots$$

is exact, and thus one can splice the complexes $\mathbf{P}$ and $\mathbf{Q}^* = \operatorname{Hom}_R(\mathbf{Q}, R)$ together to obtain an acyclic complex of free modules:

$$\mathbf{P}|\mathbf{Q}^* : \quad \cdots \longrightarrow P_2 \xrightarrow{d_2^{\mathbf{P}}} P_1 \xrightarrow{d_1^{\mathbf{P}}} P_0 \xrightarrow{\eta^* \circ \pi} Q_0^* \xrightarrow{(d_1^{\mathbf{Q}})^*} Q_1^* \xrightarrow{(d_2^{\mathbf{Q}})^*} Q_2^* \longrightarrow \cdots$$

with the convention that $(\mathbf{P}|\mathbf{Q}^*)_i = P_i$ for $i \geq 0$, and $(\mathbf{P}|\mathbf{Q}^*)_i = Q^*_{-i-1}$ for $i < 0$. This complex is minimal whenever $\mathbf{P}$ and $\mathbf{Q}$ are chosen minimal and M has no non-zero free direct summand. It is non-trivial if M is non-zero.

Definition. We say that the complex $\mathbf{A}$ is *semi-dualized* if there exists an integer s and an R-module M with $\operatorname{Ext}_R^i(M, R) = 0$ for all $i > 0$ such that $\mathbf{A}$ is isomorphic to $\mathbf{\Sigma}^s(\mathbf{P}|\mathbf{Q}^*)$, where the complex $\mathbf{P}|\mathbf{Q}^*$ is defined as in Construction 1.1, with $\mathbf{P}$ a free resolution of M^* and $\mathbf{Q}$ a free resolution of M.

The next lemma gives a useful characterization of semi-dualized complexes.

1.2. **Lemma.** *An acyclic complex $\mathbf{A}$ of free R-modules is semi-dualized if and only if there exists an integer c such that $\mathrm{H}_i(\mathbf{A}^*) = 0$ for all $i > c$.*

Proof. If $\mathbf{A}$ is semi-dualized, then $\mathbf{A} \cong \mathbf{\Sigma}^s(\mathbf{P}|\mathbf{Q}^*)$ for some s, with $\mathbf{P}|\mathbf{Q}^*$ as in 1.1. One has

$$\mathrm{H}_i(\mathbf{A}^*) \cong \mathrm{H}_{i+s-1}(\mathbf{Q}^{**}) \cong \mathrm{H}_{i+s-1}(\mathbf{Q}) = 0 \quad \text{for all } i > -s.$$

Assume now that $\mathbf{A}$ satisfies $\mathrm{H}_i(\mathbf{A}^*) = 0$ for all $i > c$. Let s be any integer satisfying $s \geq c$ and set $M = \Omega^s(\mathbf{A}^*)$. Then $\mathbf{Q} = \mathbf{\Sigma}^{-s}(\mathbf{A}^*)_{\geqslant s}$ is a free resolution of M. Since $\mathbf{A}$ is exact, we have $\mathrm{H}_i(\mathbf{Q}^*) = 0$ for all $i < 0$, hence $\operatorname{Ext}_R^i(M, R) = 0$ for all $i > 0$. Note that we have $M^* = \Omega^{-s+1}\mathbf{A}$, and hence $\mathbf{P} = \mathbf{\Sigma}^{s-1}\mathbf{A}_{\geqslant -s+1}$ is a free resolution of M^*, and thus $\mathbf{A} = \mathbf{\Sigma}^{-s+1}(\mathbf{P}|\mathbf{Q}^*)$. $\square$

An acyclic complex of free modules $\mathbf{A}$ is said to be *totally acyclic* if $\mathrm{H}(\mathbf{A}^*) = 0$. A finitely generated R-module M is said to be *totally reflexive* (or, equivalently, to have *G-dimension zero*) if the following conditions hold:

(1) the natural evaluation map $M \to M^{**}$ is an isomorphism;

(2) $\operatorname{Ext}^i_R(M,R)=0$ for all $i>0$;
(3) $\operatorname{Ext}^i_R(M^*,R)=0$ for all $i>0$.

A finitely generated R-module M is said to have *Gorenstein dimension* g, denoted $\operatorname{G-dim}_R M=g$, if g is the smallest integer such that there exists an exact sequence $0\to G_g\to\cdots\to G_1\to G_0\to M\to 0$ with G_i totally reflexive modules. If no such integer exists, then $\operatorname{G-dim}_R M=\infty$. The "Auslander-Buchsbaum formula for G-dimension" states that if $\operatorname{G-dim}_R M<\infty$, then in fact $\operatorname{G-dim}_R M=\operatorname{depth} R-\operatorname{depth} M$.

It is clear from the definition that an R-module M is totally reflexive if and only if $M\cong\Omega^0\mathbf{A}$ for some totally acyclic complex of free modules $\mathbf{A}$. We collect below some variations on this fact, as suited for our purposes.

1.3. **Lemma.** *Let $\mathbf{A}$ be an acyclic complex of finitely generated free R-modules.*

(1) *$\mathbf{A}$ is totally acyclic if and only if $\Omega^i\mathbf{A}$ is totally reflexive for all $i\in\mathbb{Z}$.*
(2) *If $\operatorname{G-dim}_R(\Omega^i\mathbf{A})<\infty$ for some i, then $\mathbf{A}$ is totally acyclic.*
(3) *If $\mathbf{A}$ is totally acyclic, then it is semi-dualized, but not conversely.*

Proof. (1) See Avramov and Martsinkovsky [4, Lem. 2.4].

(2) It is a known fact (see Auslander and Bridger [1]) that, in a short exact sequence, if two modules have finite G-dimension, so does the third. A recursive use of the short exact sequences:

$$0\to\Omega^j\mathbf{A}\to A_j\to\Omega^{j-1}\mathbf{A}\to 0$$

gives then $\operatorname{G-dim}_R(\Omega^j\mathbf{A})<\infty$ for all j. The Auslander-Buchsbaum formula for G-dimension and the depth lemma then give $\operatorname{G-dim}_R(\Omega^j\mathbf{A})=0$ for all j, and (1) shows that $\mathbf{A}$ is totally acyclic.

(3) The first part is given by Lemma 1.2. To see that the converse does not hold: Jorgensen and Şega [8] constructed minimal acyclic complexes of finitely generated free R-modules $\mathbf{C}$ with the property that $\operatorname{H}_i(\mathbf{C}^*)=0$ if and only if $i\geq 1$. Moreover, the complexes $\mathbf{C}$ in *loc. cit.* are semi-dualized. Thus not all semi-dualized acyclic complexes of free modules are totally acyclic. □

1.4. *Remark.* If R is Gorenstein, then any finitely generated R-module M satisfies $\operatorname{G-dim}_R M<\infty$. In consequence, Lemma 1.3(2) shows that the question posed in the introduction has a positive answer for all Gorenstein rings. In fact, Iyengar and Krause [7] show that Gorenstein rings are characterized by every acyclic complex of projectives being totally acyclic.

The next results indicate other classes of rings for which the question in the introduction has a positive answer.

1.5. **Proposition.** *Let $(R,\mathfrak{m})$ be a local ring which is not Gorenstein and satisfies one of the following conditions:*

(1) *R is Golod.*
(2) *R is Cohen-Macaulay and $\mathfrak{m}^2\subseteq(\boldsymbol{x})$ for some system of parameters $\boldsymbol{x}$.*
(3) $\operatorname{codim} R\leq 2$.

Then there exists no non-trivial minimal acyclic complex of finitely generated free R-modules.

1.6. **Proposition.** *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring with* $\operatorname{codim} R=3$ *and $\mathfrak{m}^3\subseteq(\boldsymbol{x})$ for some system of parameters $\boldsymbol{x}$. Then every minimal acyclic complex of free R-modules is totally acyclic.*

Together the two propositions give the result mentioned in the introduction:

1.7. **Corollary.** *Let R be a local ring of codimension c and Loewy length ℓ. Then every minimal acyclic complex of free modules is totally acyclic, and hence is semi-dualized, provided one of the following holds:*

(1) $c \leq 2$,
(2) *R is Cohen-Macaulay and $\ell \leq 2$, or*
(3) *R is Cohen-Macaulay and $c = \ell = 3$* □

We prepare next for the proofs.

1.8. **Betti numbers.** If M is a finitely generated R-module, we denote by $\beta_i^R(M)$ the ith *Betti number of* M, defined to be the integer $\operatorname{rank}_k \operatorname{Tor}_i^R(M,k)$; it is equal to the rank of the ith free module in a minimal free resolution of M.

Assume that the local ring R satisfies the following property:

($*$) There exists an integer d such that for every non-zero finitely generated R-module N with $\operatorname{pd}_R N = \infty$ the sequence $\{\beta_i^R(N)\}_{i\geq d}$ is strictly increasing.

Let $\mathbf{A}$ be a minimal acyclic complex of finitely generated free R-modules. Clearly,

$$\beta_{j-i}^R(\Omega^i\mathbf{A}) = \operatorname{rank}_R A_j \quad \text{for all } j \text{ and } i \text{ with } j \geq i\,. \tag{1.8.1}$$

If $\mathbf{A}$ is non-trivial, then Nakayama's lemma and the Auslander-Buchsbaum-Serre formula show that $\operatorname{pd}_R \Omega^i\mathbf{A} = \infty$ for all i. Then property ($*$) and (1.8.1) give that $\operatorname{rank}_R A_i > \operatorname{rank}_R A_{i-1}$ for all i, which is impossible. Hence there are no non-trivial minimal acyclic complexes of finitely generated free modules over a ring R satisfying ($*$).

1.9. **Lemma.** *If $\mathbf{A}$ is a complex of finitely generated free R-modules and $\boldsymbol{x}$ is an R-regular sequence, then the following hold:*

(1) *$(\mathbf{A}/\boldsymbol{x}\mathbf{A})^*$ and $\mathbf{A}^*/\boldsymbol{x}\mathbf{A}^*$ are isomorphic complexes of free $R/\boldsymbol{x}$-modules.*
(2) *$\mathbf{A}$ is acyclic if and only if $\mathbf{A}/\boldsymbol{x}\mathbf{A}$ is acyclic.*
(3) *$\mathbf{A}$ is totally acyclic if and only if $\mathbf{A}/\boldsymbol{x}\mathbf{A}$ is totally acyclic.*
(4) *$\mathbf{A}$ is semi-dualized if and only if $\mathbf{A}/\boldsymbol{x}\mathbf{A}$ is semi-dualized.*

Proof. (1) is straightforward and left to the reader. To prove the remaining statements we may assume that $\boldsymbol{x}$ consists of a single regular element x. Note that there exists an exact sequence of complexes $0 \to \mathbf{A} \xrightarrow{x} \mathbf{A} \to \mathbf{A}/x\mathbf{A} \to 0$ which gives rise in homology to the exact sequence

$$\cdots \to \mathrm{H}_i(\mathbf{A}) \xrightarrow{x} \mathrm{H}_i(\mathbf{A}) \to \mathrm{H}_i(\mathbf{A}/x\mathbf{A}) \to \mathrm{H}_{i-1}(\mathbf{A}) \xrightarrow{x} \cdots$$

Obviously $\mathrm{H}_i(\mathbf{A}) = 0 = \mathrm{H}_{i-1}(\mathbf{A})$ implies $\mathrm{H}_i(\mathbf{A}/x\mathbf{A}) = 0$. Also, since $\mathrm{H}_i(\mathbf{A})$ is finitely generated for each i, Nakayama's lemma gives that if $\mathrm{H}_i(\mathbf{A}/x\mathbf{A}) = 0$, then $\mathrm{H}_i(\mathbf{A}) = 0$. This proves (2). For (3), one may use parts (1) and (2) and for (4) one needs to use in addition Lemma 1.2. □

Proof of Proposition 1.5. (1) When R is Golod, Peeva shows in [10, Proposition 5] that R satisfies condition ($*$) in 1.8, hence R admits no non-trivial minimal acyclic complex of finitely generated free modules.

Under the assumption in (2), Lemma 1.9 shows that we may replace R with $R/\boldsymbol{x}R$, hence we may assume $\mathfrak{m}^2 = 0$. Let N be a finitely generated R-module and let ∂ denote the differential in a minimal free resolution of N. Since $\mathfrak{m}\operatorname{Im}\partial_i = 0$, $\operatorname{Im}\partial_i$ is a vector space over k, and it follows that $\beta_i^R(N) = e\beta_{i-1}^R(N)$ for all $i > 1$, where $e = \operatorname{edim} R$ is the embedding dimension of R. Since R is not Gorenstein, we have $e > 1$, hence R satisfies the condition $(*)$.

(3) In this case, Scheja [9] proves that the ring R is either a complete intersection, or a Golod ring. Since it is assumed that R is not Gorenstein, we conclude that R is Golod. The result is thus contained in (1). $\square$

Proof of Proposition 1.6. By 1.4, we may assume that R is not Gorenstein. Factoring out $\boldsymbol{x}$, in view of Lemma 1.9, we can assume that R is a codimension 3 ring with $\mathfrak{m}^3 = 0$. Then Theorem A of [6] gives that the Poincaré series of R satisfies $P_k^R(t) = (1 - 3t + 2t^2)^{-1}$, and thus has a pole of order 1 at $t = 1$. A result of Avramov [2, Theorem 3.1] shows that the ring R has an embedded deformation, that is, there exists a local ring $(S, \mathfrak{n})$ and a regular element $y \in \mathfrak{n}^2$ such that $R = S/(y)$. Furthermore, we have $\operatorname{codim} S = 2$, hence, according to Scheja [9], the ring S is either a complete intersection, or a Golod ring. Since we assumed that R is not Gorenstein, we conclude that S is a Golod ring.

If M is a finitely generated R-module, then one has an exact sequence (see [5, Chap. XVI, §5, Case 1.]):

$$\begin{aligned} \cdots \to \operatorname{Tor}_{n-1}^R(M, k) \to \operatorname{Tor}_n^S(M, k) \to \operatorname{Tor}_n^R(M, k) \to \\ \to \operatorname{Tor}_{n-2}^R(M, k) \to \operatorname{Tor}_{n-1}^S(M, k) \to \dots \end{aligned}$$

which gives an inequality of Betti numbers:

$$\beta_n^S(M) \le \beta_n^R(M) + \beta_{n-1}^R(M) \quad \text{for all } n \ge 1. \tag{1.9.1}$$

Let $\mathbf{A}$ be a minimal acyclic complex of finitely generated free R-modules. By [6, Theorem B] there exists $\chi \in \mathbb{Z} \cup \{\infty\}$ and a nonnegative integer c such that $\operatorname{rank}_R A_i = c$ for all $i < \chi$.

Let j be an integer with $j < \chi - (2c + 7)$. Set $M = \Omega^j \mathbf{A}$. Using (1.8.1), we have $\beta_n^R(M) = c$ for all $n \le 2c + 7$.

The inequality (1.9.1) gives

$$\beta_n^S(M) \le 2c \quad \text{for all } n \text{ with } 1 \le n \le 2c + 7 \tag{1.9.2}$$

By [10, Proposition 4], there exists a constant $B > 1$ such that

$$\beta_{n+1}^S(M) \ge B\beta_n^S(M) \quad \text{for all } n \ge 2 \operatorname{edim} R$$

Assuming that $\operatorname{pd}_S(M) = \infty$, then one has that $\beta_{n+1}^S(M) > \beta_n^S(M)$ for all $n \ge 6$. Combined with (1.9.2) this yields a contradiction. Hence $\operatorname{pd}_S(M) < \infty$.

Since $R = S/(y)$ and y is regular on S, we see that M has finite complete intersection dimension. (See Avramov, Gasharov and Peeva [3] for the definition of CI-dimension.) Thus by [3, 1.4] we have $\text{G-dim}_R(M) < \infty$, hence Lemma 1.3(2) shows that $\mathbf{A}$ is totally acyclic. $\square$

2. Minimal acyclic complexes of free modules which are not semi-dualized

In this section we show that the question raised in the introduction has a negative answer in general. More precisely, we construct minimal acyclic complexes of free modules over codimension five local rings $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ which are not semi-dualized. We then extend these to such examples over Cohen-Macaulay rings R where any choice of $\operatorname{codim} R \ge 6$ and $\ell\ell(R) \ge \operatorname{codim} R - 2$ is allowed.

2.1. Let k be a field and $\alpha \in k$ be non-zero. Consider the quotient ring

$$R = k[X_1, X_2, X_3, X_4, X_5]/I,$$

where the X_i are indeterminates (each of degree one), and I is the ideal generated by the following 11 homogeneous quadratic relations:

$$X_1^2,\ X_4^2,\ X_2X_3,\ \alpha X_1X_2\ + X_2X_4,\ X_1X_3 + X_3X_4,$$
$$X_2^2,\ X_2X_5 - X_1X_3,\ X_3^2 - X_1X_5,\ X_4X_5,\ X_5^2, X_3X_5\,.$$

As a vector space over k, R has a basis consisting of the following 10 elements:

$$1,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5,$$

where x_i denote the residue classes of X_i modulo I. Since I is generated by homogeneous elements, R is graded, and has Hilbert series $1 + 5t + 4t^2$. Moreover R has codimension five, and it is local with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_5)$ satisfying $\mathfrak{m}^3 = 0$.

For each integer $i \in \mathbb{Z}$ we let $d_i \colon R^2 \to R^2$ denote the map given with respect to the standard basis of R^2 by the matrix

$$\begin{pmatrix} x_1 & \alpha^i x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Consider the sequence of homomorphisms:

$$\mathbf{A}: \quad \cdots \to R^2 \xrightarrow{d_{i+1}} R^2 \xrightarrow{d_i} R^2 \xrightarrow{d_{i-1}} R^2 \to \cdots .$$

2.2. **Theorem.** *The sequence* $\mathbf{A}$ *is a minimal acyclic complex of free* R*-modules with* $\mathrm{H}_i(\mathbf{A}^*) \neq 0$ *for all* $i \in \mathbb{Z}$.

Lemma 1.2 gives then:

2.3. **Corollary.** *The minimal acyclic complex* $\mathbf{A}$ *is not semi-dualized.* □

2.4. *Remark.* When $\alpha \in k$ is an element of infinite multiplicative order, the complex $\mathbf{A}$ is non-periodic. When α has multiplicative order s for some integer $s > 0$, one has that $\mathbf{A}$ is periodic of period s.

2.5. *Remark.* Christensen and Veliche ask in [6, 3.5] whether every acyclic complex of free modules $\mathbf{C}$ with $\{\operatorname{rank} \mathbf{C}_i\}$ constant, over a local ring with $\mathfrak{m}^3 = 0$, is necessarily totally acyclic. Theorem 2.2 gives a negative answer to this question as well.

Proof of Theorem 2.2. Using the defining relations of R, one can easily show that $d_i d_{i+1} = 0$ for all i, hence $\mathbf{A}$ is a complex.

We let (a, b) denote an element of R^2 written in the standard basis of R^2 as a free R-module. For each i, the k-vector space $\operatorname{Im} d_i$ is generated by the elements:

$$\begin{aligned}
d_i(1,0) &= (x_1, x_3) & d_i(x_5,0) &= (x_1x_5, 0)\\
d_i(0,1) &= (\alpha^i x_2, x_4) & d_i(0,x_1) &= (\alpha^i x_1x_2, x_1x_4)\\
d_i(x_1,0) &= (0, x_1x_3) & d_i(0,x_2) &= (0, -\alpha x_1x_2)\\
d_i(x_2,0) &= (x_1x_2, 0) & d_i(0,x_3) &= (0, -x_1x_3)\\
d_i(x_3,0) &= (x_1x_3, x_1x_5) & d_i(0,x_4) &= (-\alpha^{i+1} x_1x_2, 0)\\
d_i(x_4,0) &= (x_1x_4, -x_1x_3) & d_i(0,x_5) &= (\alpha^i x_1x_3, 0)
\end{aligned}$$

Excluding $d_i(0,x_3)$ and $d_i(0,x_4)$, the above equations provide 10 linearly independent elements in $\operatorname{Im} d_i$. Thus $\operatorname{rank}_k(\operatorname{Im} d_i)=10$ for all i. Since

$$\operatorname{rank}_k \operatorname{Ker} d_i + \operatorname{rank}_k \operatorname{Im} d_i = \operatorname{rank}_k R^2 = 20$$

we have $\dim \operatorname{Ker} d_i = 10$ for all i. Thus, $\operatorname{Im} d_{i+1} = \operatorname{Ker} d_i$ for all i, so that $\mathbf{A}$ is acyclic.

To prove $\mathrm{H}_i(\mathbf{A}^*)\neq 0$, we have that $d_i^* = (d_i)^*\colon R^2 \to R^2$ is represented with respect to the standard basis of R^2 by the matrix

$$\begin{pmatrix} x_1 & x_3 \\ \alpha^i x_2 & x_4 \end{pmatrix}.$$

For each i, the vector space $\operatorname{Im} d_i^*$ is generated by the following elements

$$\begin{aligned}
d_i^*(1,0) &= (x_1,\alpha^i x_2) & d_i^*(x_5,0) &= (x_1x_5,\alpha^i x_1x_3)\\
d_i^*(0,1) &= (x_3,x_4) & d_i^*(0,x_1) &= (x_1x_3,x_1x_4)\\
d_i^*(x_1,0) &= (0,\alpha^i x_1x_2) & d_i^*(0,x_2) &= (0,-\alpha x_1x_2)\\
d_i^*(x_2,0) &= (x_1x_2,0) & d_i^*(0,x_3) &= (x_1x_5,-x_1x_3)\\
d_i^*(x_3,0) &= (x_1x_3,0) & d_i^*(0,x_4) &= (-x_1x_3,0)\\
d_i^*(x_4,0) &= (x_1x_4,-\alpha^{i+1}x_1x_2) & d_i^*(0,x_5) &= (0,0)
\end{aligned}$$

Excluding $d_i^*(0,x_2)$, $d_i^*(0,x_4)$, and $d_i^*(0,x_5)$ which are redundant, we have only 9 linearly independent elements in $\operatorname{Im} d_i^*$, hence $\operatorname{rank}_k \operatorname{Im} d_i^* = 9$ for every i. It follows that $\operatorname{rank}_k \operatorname{Ker} d_i^* = 11$ for all i, hence $\mathrm{H}_i(\mathbf{A}^*)\neq 0$. $\square$

One can easily get examples of minimal acyclic complexes which are not semidualized over local rings of any codimension larger than five as follows.

Let $n\geq 1$ be an integer, and $y_1,\dots,y_n$ be indeterminates over k. Define R_n to be the local ring obtained by localizing $R\otimes_k k[y_1,\dots,y_n]$ at the maximal ideal

$$\mathfrak{m}_n = (x_i\otimes 1, 1\otimes y_j \mid 1\leq i\leq 5, 1\leq j\leq n)$$

Now let $\mathbf{A}_n$ denote the sequence

$$\cdots \to R_n^2 \xrightarrow{d_{i+1}^n} R_n^2 \xrightarrow{d_i^n} R_n^2 \xrightarrow{d_{i-1}^n} R_n^2 \to \cdots,$$

where d_i^n denotes the map $d_i\otimes_k k[y_1,\dots,y_n]$ localized at $\mathfrak{m}_n$.

Let $p_1,\dots,p_n$ be positive integers ≥ 2, set $\ell = \sum_{i=1}^n (p_i-1)+3$, and consider the R_n-sequence $\boldsymbol{y} = 1\otimes y_1^{p_1},\dots,1\otimes y_n^{p_n}$.

2.6. **Corollary.** *The ring R_n is a local Cohen-Macaulay ring with* $\operatorname{codim}(R_n)=5$*,* $\dim(R_n)=n$ *and* $\ell\ell(R_n)=3$*, and $S_n = R_n/(\boldsymbol{y})$ is an artinian local with* $\operatorname{codim}(S_n)=n+5$ *and* $\ell\ell(S_n)=\ell$*, such that the following hold:*

1. *$\mathbf{A}_n$ is a minimal acyclic complex of free R_n-modules which is not semidualized.*
2. *$\mathbf{A}_n/\boldsymbol{y}\mathbf{A}_n$ is a minimal acyclic complex of free S_n-modules which is not semidualized.*

Proof. The statements about R_n and S_n are clear.

Since $R\to R\otimes_k k[y_1,\dots,y_n]$ is a faithfully flat embedding of rings, the complex $\mathbf{A}\otimes_k k[y_1\dots,y_n]$ stays exact, and then too after localizing. Thus $\mathbf{A}_n$ is acyclic (and obviously minimal).

We have $\mathrm{H}_i(\mathbf{A}^*)\otimes_k k[y_1,\dots,y_n]\cong \mathrm{H}_i(\mathbf{A}^*\otimes_k k[y_1,\dots,y_n])$ for all i. Moreover, $\mathbf{A}^*\otimes_k k[y_1,\dots,y_n]$ localized at $\mathfrak{m}_n$ is isomorphic to $(\mathbf{A}_n)^*$. Since $\mathrm{H}_i(\mathbf{A}^*)\neq 0$ for all i by Theorem 2.2, it follows that $\mathrm{H}_i((\mathbf{A}_n)^*)\neq 0$ for all $i\in\mathbb{Z}$. Hence $\mathbf{A}_n$ is not semi-dualized by Lemma 1.2. The statement about $\mathbf{A}_n/\boldsymbol{y}\mathbf{A}_n$ follows from Lemma 1.9. $\square$

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Meri T. Hughes, Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019

E-mail address: `meri@uta.edu`

David A. Jorgensen, Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019

E-mail address: `djorgens@uta.edu`

Liana M. Şega, Department of Mathematics and Statistics, University of Missouri-Kansas City, Kansas City, MO 64110

E-mail address: `segal@umkc.edu`